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## Exact solutions for the mean field plus state-dependent pairing Hamiltonian

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**Abstract.** The nuclear mean field plus state-dependent pairing problem is studied by using the Bethe ansatz method. It is shown that the Hamiltonian can be expanded in terms of generators of the infinite-dimensional Lie algebra  $\widehat{SU}(2)$  without central extension. Exact solutions for excited energies and the corresponding eigenstates of the nuclear pairing problem are derived through the Bethe ansatz equations.

Exactly solvable models are of interest in both physics and mathematics. The importance of an exactly solvable theory is that it can be used to test the validity of various approximate methods for solving the same problem. The results are also useful for probing the nature of the solution, especially its asymptotic behaviour. A large number of quantum integrable and exactly solvable models have been found by using the inverse scattering method [1–4]. The main idea of this method rests on the use of a special associative algebra known as a Yang–Baxter construction [5, 6]. Up to now, most efforts have focused on quantum spin systems with nearest-neighbour interactions [7], vertex models in statistical mechanics [8, 9], one-dimensional Schrödinger equations [5, 10] and Hubbard models [11, 12].

There are also some many-body problems in nuclear physics that are exactly solvable, for example, exact solutions exist for the pure pairing problem ( $SU(2)$  quasi-spin [13]) and a quadrupole–quadrupole interaction ( $SU(3)$  shell model [14]). The three limits of the interacting boson model are other examples [15]. To obtain exact solutions for theories that reach beyond exactly solvable models requires matrix diagonalizations. In most cases, the dimension of the Fock space required in the diagonalization is very large, which makes computation infeasible. Hence, many approximation methods are adopted. For example, for the case of generalized pairing the BCS and Hartree–Fock–Bogolyubov (HFB) approximations are often used, sometimes in conjunction with correction terms evaluated within the random-phase approximation (RPA). However, in this case there are some potentially serious pitfalls. First of all, not only is the number of nucleons in a nucleus of interest often small, the number of valence particles which dominates the behaviour of its low-lying states is usually too few to justify the underlying assumptions of the approximation. As a result, particle-number-non-conservation effects can enter and this can lead to other serious difficulties, such as spurious states, non-orthogonal solutions, etc. Another problem with an approximate treatment

of pairing in nuclei is related to the fact that both the BCS and the HFB approximations break down for a very important class of physical situations. The usual remedy in terms of particle number projection techniques complicates the algorithms considerably without yielding a better description of the more highly excited part of the spectrum. It is for these reasons that particle-number-conserving methods, even if only approximate, are important for probing the true nature of pairing effects in nuclei. The first attempt to find exact solutions of the non-degenerate nuclear pairing problem was made by Richardson who considered the equal pairing strength approximation which has an orbit-independent solution [16–18]. Recently, it has been shown that a mean field plus separable pairing interaction, which includes the equal strength pairing case discussed by Richardson as a special limit, is exactly solvable by using infinite-dimensional algebraic methods [19–21]. However, exact solutions to the general pairing problem beyond matrix diagonalization still needs to be explored. In this letter, it will be shown that the nuclear mean field plus state-dependent pairing interaction Hamiltonian can also be exactly solved by using an infinite-dimensional algebraic method.

The general pairing Hamiltonian for spherical nuclei can be written as

$$\hat{H} = \sum_j \varepsilon_j \Omega_j + 2 \sum_j \varepsilon_j S^0(j) - \sum_{jj'} c_{jj'} S^+(j) S^-(j') \quad (1)$$

where  $\varepsilon_j$  are single-particle energies and  $S^\pm(j)$  and  $S^0(j)$  are the pairing operators for a single- $j$  shell defined by

$$\begin{aligned} S^+(j) &= \sum_{m>0} (-)^{j-m} a_{jm}^\dagger a_{j-m}^\dagger & S^-(j) &= \sum_{m>0} (-)^{j-m} a_{j-m} a_{jm} \\ S^0(j) &= \frac{1}{2} \sum_{m>0} (a_{jm}^\dagger a_{jm} + a_{j-m}^\dagger a_{j-m} - 1) = \frac{1}{2} (\hat{N}_j - \Omega_j). \end{aligned} \quad (2)$$

In (2),  $\Omega_j \equiv j + \frac{1}{2}$  is the maximum number of pairs in the  $j$ th shell,  $a_{jm}^\dagger$  and  $a_{jm}$  with  $m = -j, -j+1, \dots, j$ , are nucleon creation and annihilation operators, respectively,  $\hat{N}_j$  is the particle number operator for the  $j$ th shell and  $c_{jj'}$  in (1) is the strength of the pairing interaction between the  $j$  and  $j'$  shells. If the number of the orbits is  $p$ , it is easy to verify that the operators given in (2) generate a direct sum of Lie algebra:  $\oplus_{i=1}^p SU_i(2)$ .

It is clear that there are two sets of parameters ( $\{\varepsilon_j\}$  and  $\{c_{jj'}\}$ ) in equation (1). In non-degenerate cases, the  $\varepsilon_j$  are real numbers that are not equal to each other. In this case, one can assume that the parameters  $c_{jj'}$  can be expanded in terms of  $\varepsilon_j$  and  $\varepsilon_{j'}$  as

$$c_{jj'} = \sum_{mn} g_{mn} \varepsilon_j^m \varepsilon_{j'}^n \quad (3)$$

where  $\{g_{mn}\}$  is a set of parameters to be determined according to equation (3). Hence, similar to the separable pairing case [19–21] we can introduce the operators  $\{S_n^\mu; \mu = 0, +, -; n = 0, 1, 2, \dots\}$  with

$$S_n^+ = \sum_j \varepsilon_j^n S^+(j) \quad S_n^- = \sum_j \varepsilon_j^n S^-(j) \quad S_n^0 = \sum_j \varepsilon_j^n S^0(j). \quad (4)$$

The operators  $\{S_n^\mu\}$ , which form a half-positive infinite-dimensional Lie algebra  $\widehat{SU}(2)$  without central extension, satisfy the following commutation relations:

$$[S_m^+, S_n^-] = 2S_{m+n}^0 \quad [S_m^0, S_n^\pm] = \pm S_{m+n}^\pm. \quad (5)$$

Using these  $\widehat{SU(2)}$  generators, one can rewrite the Hamiltonian (1) as

$$\hat{H} = \sum_j \varepsilon_j \Omega_j + 2S_1^0 - \sum_{mn} g_{mn} S_m^+ S_n^- \tag{6}$$

In order to diagonalize the Hamiltonian (6), we use the following Bethe ansatz [22] wavefunction:

$$|k; \zeta\rangle = \mathcal{N} S^+(x_1^{(\zeta)}) S^+(x_2^{(\zeta)}) \cdots S^+(x_k^{(\zeta)}) |0\rangle \tag{7}$$

where  $\mathcal{N}$  is a normalization constant,  $\zeta$  is an additional quantum number used to distinguish different eigenstates with the same number of pairs  $k$ ,  $|0\rangle$  is the pairing vacuum state defined by

$$S^-(j)|0\rangle = 0 \quad \text{for all } j \tag{8}$$

and

$$S^+(x_r^{(\zeta)}) = \sum_m a_m S_m^+(x_r^{(\zeta)}) \tag{9}$$

in which  $\{a_m\}$  and  $\{x_r^{(\zeta)}\}$  are two sets of  $c$ -numbers to be determined and

$$S_m^+(x_r^{(\zeta)}) = \sum_j \frac{\varepsilon_j^m}{1 - \varepsilon_j x_r^{(\zeta)}} S_j^+ \tag{10}$$

In solving the eigenvalue equation

$$\hat{H}|k; \zeta\rangle = E_k^{(\zeta)}|k; \zeta\rangle \tag{11}$$

we observe that like the separable pairing case [20, 21], auxiliary conditions are necessary to cancel of the so-called unwanted terms. It can be verified that the auxiliary conditions (or the so-called additional *ad hoc* Bethe ansatz) can be chosen as

$$\sum_v a_v \varepsilon_j^v G_{ij} = \sum_s \frac{c_s^{(i)}}{1 - \varepsilon_j z_s^{(i)}} \tag{12}$$

where  $\{c_s^{(i)}\}$  and  $\{z_s^{(i)}\}$  are another two sets of unknown  $c$ -numbers to be determined and

$$G_{nj} = \sum_m g_{nm} \varepsilon_j^m \tag{13}$$

Using (11), (12) and commutation relations (5), one can prove that

$$E_k^{(\zeta)} = \sum_{i=1}^k \frac{2}{x_i^{(\zeta)}} \tag{14}$$

Furthermore, the  $c$ -numbers  $\{a_m\}$  ( $m = 0, 1, \dots, p - 1$ ),  $\{x_r^{(\zeta)}\}$  ( $r = 1, 2, \dots, k$ ),  $\{c_s^{(i)}\}$  and  $\{z_s^{(i)}\}$  ( $0 \leq i, s \leq p - 1$ ) must satisfy

$$\frac{a_i}{x_\mu^{(\zeta)}} = \Lambda_i(x_\mu^{(\zeta)}) + \sum_{v \neq \mu} \frac{x_v^{(\zeta)}}{x_v^{(\zeta)} - x_\mu^{(\zeta)}} \mathcal{A}_i(x_v^{(\zeta)}) \tag{15a}$$

and

$$\sum_{r>q} \sum_s \frac{c_s^{(i)} (z_s^{(i)})^2}{(1 - \varepsilon_j z_s^{(i)}) (z_s^{(i)} - x_r^{(\zeta)}) (z_s^{(i)} - x_q^{(\zeta)})} = \sum_{r>q} \frac{a_i}{(1 - \varepsilon_j x_r^{(\zeta)}) (1 - \varepsilon_j x_q^{(\zeta)})} \tag{15b}$$

where

$$\Lambda_m(x) = \sum_{n\mu} \langle S_{\mu+n}^0(x) \rangle a_\mu g_{mn} \tag{16a}$$

with

$$\langle S_v^0(x) \rangle = \frac{1}{2} \sum_j \frac{\varepsilon_j^v(\tau - \Omega_j)}{1 - \varepsilon_j x} \tag{16b}$$

where  $\tau = \sum_j \tau_j$  is the seniority quantum number of the pairing vacuum state defined by (8) and

$$\mathcal{A}_i(x) = a_i - \sum_s \frac{c_s^{(i)} x}{z_s^{(i)} - x}. \tag{16c}$$

It can be easily seen that equations (12) and (15b) give  $2p \times p$  relations which are necessary and sufficient conditions to express  $\{c_s^{(i)}\}$  and  $\{z_s^{(i)}\}$  as functions of  $a_m$  and  $\{x_v^{(\zeta)}\}$ . The remaining problem is to obtain roots  $a_m$  and  $\{x_i^{(\zeta)}\}$  from equation (15a). It is obvious that the Bethe ansatz equation (15a) has  $S_k$  symmetry. Any permutation among different roots  $x_i^{(\zeta)}$  for  $i = 1, 2, \dots, k$  in (15a) is invariant, which is also a required condition for the Bethe ansatz wavefunction (7). It can be proved from (15a) that the amplitudes  $a_m$  for  $m = 0, 1, \dots, p - 1$  in the expansion (11) can be expressed as

$$a_m = \frac{1}{k} \left\{ \sum_\mu \Lambda_m(x_\mu^{(\zeta)}) + \sum_{r>q} \frac{x_r^{(\zeta)} x_q^{(\zeta)}}{x_r^{(\zeta)} - x_q^{(\zeta)}} (\mathcal{A}_m(x_r^{(\zeta)}) - \mathcal{A}_m(x_q^{(\zeta)})) \right\}. \tag{17}$$

Namely,  $a_m$  must be symmetric functions of  $\{x_1^{(\zeta)}, x_2^{(\zeta)}, \dots, x_k^{(\zeta)}\}$ . However, equation (17) gives  $p$  nonlinear transformations among  $\{a_0, a_1, \dots, a_{p-1}\}$ , which can be determined only when one of the amplitudes  $a_m$  for fixed  $m$  is known. Due to the fact that the wavefunction (7) is determined up to a normalization constant  $\mathcal{N}$ , we simply choose  $a_0 = 1$ . Other amplitudes  $\{a_1, a_2, \dots, a_{p-1}\}$  and roots  $\{x_i^{(\zeta)}\}$  for  $i = 1, 2, \dots, k$  can then be solved uniquely using equation (15a) with the auxiliary Bethe ansatz equations (12) and (15b).

As a simple example of the theory, we consider the  $J = 0$  pairing spectra of the even-even oxygen isotopes  $^{18-26}\text{O}$ . The neutron single-particle energies  $\varepsilon_j$  were taken from the energy spectra of  $^{17}\text{O}$  with  $\varepsilon_{1/2} = -3.273$  MeV,  $\varepsilon_{3/2} = 0.941$  MeV and  $\varepsilon_{5/2} = -4.143$  MeV. These values are all relative to the binding energy of  $^{16}\text{O}$ , which was taken to be zero. The two-body general pairing strengths  $c_{jj'}$  in MeV were taken from the  $J = 0$  two-body matrix elements of the universal ds-shell Hamiltonian [23] with  $c_{1/2\ 1/2} = 2.125$ ,  $c_{3/2\ 3/2} = 1.092$ ,  $c_{5/2\ 5/2} = 0.940$ ,  $c_{1/2\ 3/2} = 0.766$ ,  $c_{1/2\ 5/2} = 0.765$  and  $c_{3/2\ 5/2} = 1.301$ . Using these data, one can obtain the following symmetric  $g_{mn}$  parameters from equation (6):  $g_{00} = 2.047\ 61$  MeV,  $g_{10} = -1.206\ 21$ ,  $g_{02} = -0.353\ 204$  MeV $^{-1}$ ,  $g_{11} = 1.320\ 29$  MeV $^{-1}$ ,  $g_{12} = 0.405\ 14$  MeV $^{-2}$ ,  $g_{22} = 0.122\ 13$  MeV $^{-3}$ . The resulting excitation energies calculated from (12)–(15) are shown in table 1. The corresponding exact eigenstates can also be obtained immediately. For example, the one-pair states in this case can be expressed as

$$|k = 1, \zeta\rangle = \sum_{m=0}^2 a_m^{(\zeta)} S_m^+(x^{(\zeta)})|0\rangle \tag{18}$$

with  $a_0^{(\zeta)} = 1$  and

$$\begin{aligned} a_1^{(\zeta)} &= \frac{0.888\ 763x^{(\zeta)}(x^{(\zeta)2} + 0.339\ 93x^{(\zeta)} + 0.038\ 976)}{(x^{(\zeta)} - 2.0661)(x^{(\zeta)} + 0.228\ 87)(x^{(\zeta)} + 0.336\ 132)} \\ a_2^{(\zeta)} &= \frac{0.330\ 653x^{(\zeta)}(x^{(\zeta)2} + 0.287\ 101x^{(\zeta)} + 0.022\ 8806)}{(x^{(\zeta)} - 2.0661)(x^{(\zeta)} + 0.228\ 87)(x^{(\zeta)} + 0.336132)}. \end{aligned} \tag{19}$$

**Table 1.** Pairing excitation energies (in MeV) for even–even  $^{18-26}\text{O}$  calculated from equations (14)–(17).

$\zeta$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
0	-12.602 8	-23.149 8	-31.118 2	-37.935 1	-37.819 0	-34.768 1
1	-8.095 95	-19.270 8	-26.752 6	-29.916 3	-27.144 5	—
2	0.621 055	-11.263 5	-21.604 6	-27.948 9	-25.207 2	—
3	—	-7.631 72	-17.514 6	-18.764 6	—	—
4	—	2.408 87	-9.238 28	-16.016 4	—	—
5	—	—	-4.773 64	—	—	—

For the one-pair ground state, we have  $x^{(0)} = -0.247\,103\text{ MeV}^{-1}$  with  $a_1^{(0)} = -0.940\,131$ ,  $a_2^{(0)} = -0.283\,429$ ; and  $x^{(1)} = -0.158\,6477\text{ MeV}^{-1}$  with  $a_1^{(1)} = 0.051\,9519$ ,  $a_2^{(1)} = 0.047\,3969$  and  $x^{(2)} = 3.219\,89\text{ MeV}^{-1}$  with  $a_1^{(2)} = 2.326\,03$ ,  $a_2^{(2)} = 0.851\,362$  for the one-pair first and second excited states, respectively. It is obvious that the procedure can be applied to more realistic calculations for the nuclear pairing problem. For example, shell model results reported in [22] for some low-lying levels, spectroscopic factors and even–odd mass difference of  $^{58-65}\text{Ni}$  and in [25] for binding energies of  $^{18-26}\text{O}$  can now be derived from equations (14)–(17).

It can be verified that the building blocks  $S_m^\pm(x)$  given by (10) of the Bethe ansatz wavefunction (7) with operators

$$S_m^-(x) = (S_m^+(x))^\dagger \quad S_m^0(x) = \sum_j \frac{\varepsilon_j^m}{1 - \varepsilon_j x} S^0(j) \quad (20)$$

generate the nonlinear algebra  $\mathcal{G}(SU_2)$ , which is an infinite-dimensional extension of the Gaudin algebra given in [24]. The commutation relations of these generators are

$$\begin{aligned} [S_m^+(x), S_n^-(y)] &= \frac{2}{x-y} (xS_{m+n}^0(x) - yS_{m+n}^0(y)) \\ [S_m^0(x), S_n^\pm(y)] &= \pm \frac{1}{x-y} (xS_{m+n}^\pm(x) - yS_{m+n}^\pm(y)). \end{aligned} \quad (21)$$

In conclusion, a novel systematic infinite-dimensional algebraic method for finding exact solutions to the mean field plus the state-dependent pairing Hamiltonian is proposed. The method can be extended to other many-body problems such as Hubbard models, interacting boson systems and so on, as has already been demonstrated for the case of separable potentials [19–21, 25–27]. The strength of the method is that exact solutions of quantum many-body problems with general two-body interaction can be obtained systematically and this provides a means for solving various many-body problems and reveals new nonlinear infinite-dimensional algebraic symmetries.

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